VOL. 1 NO. 3 (2022)

ISSN: 2181-3132

YOSH TADQIQOTCHI JURNALI

2nd Sun

SOXOTSKIY-PLEMEL FORMULALARI

[1]Allanazarov Otabek Ilhomjon o'g'li

[2]Jovliyeva Kamola Dilmuratovna

[1]Termiz Davlat Universiteti Matematika (differensial tenglamalar va matematik fizika) yo`nalishi 2-bosqich magistranti

[2]Termiz Davlat Universiteti magistranti

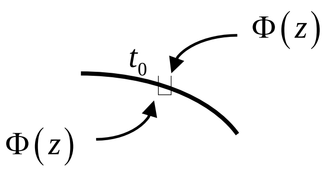
https://doi.org/10.5281/zenodo.6509517

ANNOTATSIYA: Ushbu maqolada singulyar integral tenglamalarni regulyarizatsiyalash usullaridan biri, Soxotskiy-Plemel formulalari ko'rib chiqiladi.

Kalit so`zlar: Singulyar integral tenglamalar, Soxotskiy-Plemel formulalari, regulyarizatsiyalash, kontur chegarasi, cheksizlik, tenglama.

t_0 nuqtaga turli tomondan yaqinlashganda Koshi tipidagi integralning limitini ko'rib chiqamiz.

$$\Phi_+(t_0) = \lim_{\substack{z\to t_0 \\ z\in G}} \Phi(z)$$



$$\Phi_-(t_0) = \lim_{\substack{z\to t_0 \\ z\notin \overline{G}}} \Phi(z)$$

Masalan: $\int_T \frac{dt}{t-z} = \begin{cases} 2\pi i, & z \in D \quad \Phi_+ \\ 0, & z \notin D \quad \Phi_- \\ \pi i, & z \in \mathrm{T} \quad \Phi \end{cases}$

Soxotskiy-Plemel formulalari Φ_+, Φ_- va Φ larni o'zaro bog'laydi.

Teorema: $\varphi(z) = \int_\Gamma \frac{\mu(t)-\mu(t_0)}{t-z} dt \xrightarrow[z\to t_0]{} \varphi(t_0) = \int_\Gamma \frac{\mu(t)-\mu(t_0)}{t-t_0} dt$ funksiya ($z \to t_0$, $z \in G$, $z \notin \overline{G}$ da uzluksiz)

$$\varphi(z) - \varphi(t_0) = \int_\Gamma \frac{\mu(t)-\mu(t_0)}{(t-z)(t-t_0)}(z-t_0)dt = (z-t_0)\int_\Gamma \frac{\mu(t)-\mu(t_0)}{(t-z)(t-t_0)}dt$$

I. Dastlab $z \to t_0$ yaqinlashish Γ ga urinma bo'lmagan yo'nalish bo'yicha sodir bo'lsin.

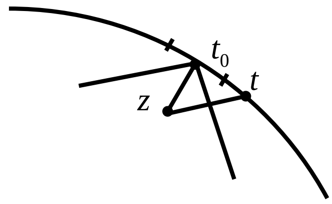
ISSN: 2181-3132
YOSH TADQIQOTCHI JURNALI

Kichik ρ belgilash kiritamiz: $T(t_0, \rho)$

$$\left|\int\limits_{\gamma_\rho}\right| \leq |z-t_0| \cdot \frac{C_0}{|z-t_0|} \int\limits_{\gamma_\rho} \left|\frac{\mu(t)-\mu(t_0)}{t-t_0}\right| dt \leq t \in \gamma_\rho$$

$$|t-z| \geq C_0 \cdot |z-t_0|$$

$$|\mu(t)-\mu(t_0)| \leq C_0 \cdot |t-t_0|^\alpha$$



(Ushbu shartdan foydalanamiz: $H_\alpha \Rightarrow$)

$$\leq C_0 \cdot 2 \int\limits_0^\rho \frac{x^\alpha}{x} dx \leq C_0 \cdot 2 \cdot \frac{x^\alpha}{\alpha}\Big|_0^\rho = \frac{2C_0}{\alpha}\rho^\alpha$$

$\left|\int_{\Gamma_\rho}\right| \leq$($|t-z| \approx |t-t_0|$ bo'lishi uchun $|z-t_0| < \varepsilon_0$ ni olamiz)

$$\leq |z-t_0| \cdot \left|\int\limits_{\Gamma_\rho} \frac{\mu(t)-\mu(t_0)}{(t-t_0)(t-z)} dt\right| \leq$$

($t \in \Gamma_\rho$, u holda $|t-z| \geq C_1 \cdot |t-t_0|$)

$$\leq \frac{|z-t_0|}{C_1} \int\limits_{\Gamma_\rho} \frac{\mu(t)-\mu(t_0)}{|t-t_0|^2} dt \leq$$

($t \in \Gamma_\rho, |t-t_0| \geq C_2 \cdot \rho, \mu - \Gamma$ da uzluksiz)

$$\Rightarrow |\mu(t)-\mu(t_0)| \leq 2\max_\Gamma \mu(t) \leq 2\mathrm{M} \leq \frac{|z-t_0|}{C_1} \cdot \frac{2\mathrm{M}}{\rho^2} \cdot |\Gamma|$$

$$\left|\int\limits_{\gamma_\rho}\right| \leq C \cdot \rho^\alpha$$

ISSN: 2181-3132

YOSH TADQIQOTCHI JURNALI

2nd Sun

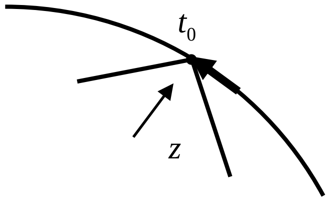
$$\left|\int_{\Gamma_\rho}\right| \leq C \cdot \frac{|z-t_0|}{\rho^2}$$

Dastlab shunchalik kichik ρ ni olamizki, $C\rho^\alpha < \frac{\varepsilon}{2}$

So'ng shunchalik kichik $|z-t_0|$ ni olamizki, $\frac{|z-t_0|}{\rho^2}C \leq \frac{\varepsilon}{2}$

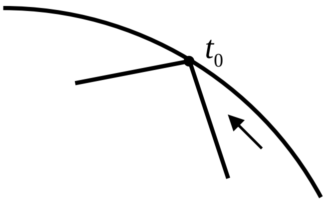
$$\left|\int_{\Gamma_\rho}\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Ya'ni, $\varphi(z) \underset{z \to t_0}{\longrightarrow} \varphi(t_0)$ urinma bo'lmagan yo'l orqali o'tadi.



II. z nuqta t_0 nuqtaga urinma bo'lmagan yo'ldan yaqinlashishidan qutilishimiz kerak.

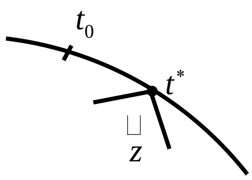
$\varphi(t_0) = \int_\Gamma \frac{\mu(t)-\mu(t_0)}{t-t_0}dt$ funksiya t_0 bo'yicha uzluksin, chunki $\mu(t) \in$ $H_\alpha \Rightarrow$



$\Rightarrow \int_\Gamma \left|\frac{\mu(t)-\mu(t_0)}{t-t_0}\right| dt \leq \int \frac{x^\alpha}{x}dx$ integral yaqinlashadi $\Rightarrow$

$\Rightarrow$(Veyershtrass teoremasiga ko'ra) $\varphi(t_0)$ - uzluksiz

$z \in G, t_0 \in \Gamma$ bo'lsin,



t^* – Γ egri chiziqda yotgan va z nuqtaga eng yaqin nuqta bo'lsin.

$$|z-t^*| = \min_{t\in\Gamma}|z-t|$$

Bu holda z – uchi t^* va qiymati $\frac{\pi}{2}$ bo'lgan burchakka tegishli.

Agar $|z-t_0| < \delta$ bo'lsa, $\begin{matrix}|z-t^*| < \delta \\ |t^*-t_0| < \delta\end{matrix} \Rightarrow \begin{matrix}|\varphi(z)-\varphi(t^*)| < \frac{\varepsilon}{2} \\ |\varphi(t^*)-\varphi(t_0)| < \frac{\varepsilon}{2}\end{matrix}$

$$|\varphi(z)-\varphi(t_0)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Koshi tipidagi integralni integrallash konturida o'rganamiz. Keyinchalik qo'lga kiritiladigan asosiy natija shundan iborat bo'ladiki, Gyolder shartini qanoatlantiradigan Koshi tipidagi zichlikli integral o'zini ikki qatlam uzluksiz zichlik potensialidek tutadi, ya'ni, konturga yaqinlashishida uzluksiz limitlarga ega bo'ladi, ammo bu limitlarning qiymatlari turli xil bo'ladi, kontur bo'yicha o'tishda skarash bo'lishi uchun. Avval lemmani ko'rib chiqaylik:

Asosiy lemma. Agar $\varphi(\tau)$ yuk Gyolder shartini qanoatlantirsa va t nuqta L konturning chegarasiga to'g'ri kelmasa,

$$\Psi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)-\varphi(t)}{\tau-z}d\tau\,,\qquad(1)$$

funksiya $z=t$ nuqtadan o'tgunicha uzluksiz funksiyadek ko'rinadi, ya'ni, z nuqta t ga yaqinlashganida funksiya aniq limit qiymatga ega bo'ladi:

$$\lim_{z\to t}\Psi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)-\varphi(t)}{\tau-z}d\tau=\Psi(t)$$

Integral tipidagi Koshi masalasini ko'rib chiqaylik:

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)}{\tau-z}d\tau\,,\qquad(2)$$

Bunda $\varphi(\tau)$ Gyolder shartini qanoatlantiradi. (2) ning qiymatini L konturda

$$\Phi(t)=\int_L\frac{\varphi(\tau)}{\tau-z}d\tau$$

deb belgilab olamiz.

Bunda integral Koshi bo'yicha asosiy qiymatga ega deb tushuniladi.

L konturni yopiq va silliq deb hisoblaylik. Agar kontur yopiq bo'lmay qolsa, uni to'liq yopiq bo'lgunicha qo'shimcha egri chiziq bilan, ushbu egri chiziqqa $\varphi(t)=0$ yuk qo'yib to'ldiramiz.

$\Phi(z)$ funksiyaning limitini konturdagi biror t nuqtada o'rganish uchun (1) funksiyani olamiz. $\Phi^+(t)$, $\Psi^+(t)$ deb $\Phi(z)$, $\Psi(z)$ analitik funksiyalarining z nuqtaning t nuqtaga L konturning ichidan, va $\Phi^-(t)$, $\Psi^-(t)$ deb esa kontur tashqarisidan yaqinlashishiga aytamiz (bu yopiq bo'lmagan konturlar uchun chap va o'ng limitlarga mos keladi). Limitga o'tish yo'nalishiga urg'u berish uchun mos ravishda $z \to t^+$ yoki $z \to t^-$ yozuvlarini ishlatamiz. Konturdagi t nuqtadagi funksiyalarning qiymatlarini shunchaki $\Phi(t), \Psi(t)$ deb belgilaymiz. $\varphi(t), \Phi(t)$ va $\Phi^\pm(t)$ lar orasidagi bog'liqlikni ko'rib chiqamiz.

$$\int_L \frac{d\tau}{\tau - z} = \begin{cases} 2\pi i, & z \in D^+ \\ 0, & z \in D^-, \\ \pi i, & z \in L \end{cases} \qquad (3)$$

Ekanligidan kelib chiqqan holda,

$$\Psi^+(t) = \lim_{z \to t^+} \left[\frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z} d\tau - \frac{\varphi(t)}{2\pi i} \int_L \frac{d\tau}{\tau - z} \right] = \Phi^+(t) - \varphi(t),$$

$$\Psi^-(t) = \lim_{z \to t^-} \left[\frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z} d\tau - \frac{\varphi(t)}{2\pi i} \int_L \frac{d\tau}{\tau - z} \right] = \Phi^-(t),$$

$$\Psi(t) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau - \frac{\varphi(t)}{2\pi i} \int_L \frac{d\tau}{\tau - t} = \Phi(t) - \frac{1}{2}\varphi(t)$$

Asosiy lemmaga ko'ra $\Psi(z)$ uzluksiz bo'lganligi sababli, ushbu tengliklarning o'ng qismlari bir xil, ya'ni,

$$\Phi^+(t) - \varphi(t) = \Phi^-(t) = \Phi(t) - \frac{1}{2}\varphi(t), \qquad (4)$$

Bundan quyidagi tengliklarga kelamiz:

$$\Phi^+(t) = \frac{1}{2}\varphi(t) + \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau,$$

$$\Phi^-(t) = -\frac{1}{2}\varphi(t) + \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau, \qquad (5)$$

(28) formulalarni ayirib va qo'shib yuborib,

$$\Phi^+(t) - \Phi^-(t) = \varphi(t), \qquad (6)$$

$$\Phi^{+}(t) + \Phi^{-}(t) = \frac{1}{\pi i}\int_{L} \frac{\varphi(\tau)}{\tau - t} d\tau\,, \qquad (7)$$

Formulalarga kelamiz. Ushbu natijalarni teorema ko'rinishida ifodalaymiz:

Teorema. L – silliq kontur (yopiq yoki yopiq emas) va $\varphi(\tau)$ – Gyolder shartini qanoatlantiruvchi kontur nuqtalarining funksiyasi bo'lsin. Bu holda Koshi tipidagi

$$\Phi(z) = \frac{1}{2\pi i}\int_{L} \frac{\varphi(\tau)}{\tau - z} d\tau$$

integral L konturning chegarasida bo'lmagan har qaysi nuqtada, konturga chapdan yaqinlashsa ham, o'ngdan yaqinlashsa ham ma'lum $\Phi^{+}(t)$ va $\Phi^{-}(t)$ limitga ega, va bu limitlar (5), (6) va (7) Soxotskiy-Plemel formulalari bo'yicha $\varphi(t)$ integralning zichligi va $\Phi(t)$ integrali orqali ifodalanadi.

Konturning chetki nuqtalari uchun Soxotskiy-Plemel formulalari.

$z \in L$ bo'lganida chetki nuqtalarida (3) integralning qiymati o'rniga quyidagi integraldan foydalanamiz:

$$\int_{L} \frac{d\tau}{\tau - z} = i\alpha$$

Bunda α – L konturdagi t chetki nuqtaga o'tkazilgan chap va o'ng urinma vektorlari orasidagi burchak.

(4) tenglik bu holda

$$\Phi^{+}(t) - \varphi(t) = \Phi^{-}(t) = \Phi(t) - \frac{\alpha}{2\pi}\varphi(t),$$

ko'rinishiga keladi va bu tenglikdan $\Phi^{+}(t)$ va $\Phi^{-}(t)$ lar uchun ifodani hosil qila olamiz:

$$\Phi^{+}(t) = \left(1 - \frac{\alpha}{2\pi}\right)\varphi(t) + \frac{1}{2\pi i}\int_{L} \frac{\varphi(\tau)}{\tau - t} d\tau\,, \qquad (8)$$

$$\Phi^{-}(t) = -\frac{\alpha}{2\pi}\varphi(t) + \frac{1}{2\pi i}\int_{L} \frac{\varphi(\tau)}{\tau - t} d\tau\,, \qquad (9)$$

Natijani quyidagi ko'rinishda yozish ham mumkin:

Teorema. Agar Koshi tipidagi integral chekli chetki nuqtalarga ega bo'lgan kontur bo'yicha olinsa, integralning limit qiymatlari mavjud va chetki bo'lmagan nuqtalarda Soxotskiy-Plemelning oddiy formulalari (5) va chetki nuqtalar uchun (8) va (9) o'rinli bo'ladi.

Ushbu isbot bevosita kvadratning nuqtalariga ta'sir etmaydi. Ammo, qo'shimcha mulohazalar orqali ko'rsatish mumkin-ki, (8) va (9) formulalar bu holatlarda ham o'rinli bo'ladi. Bunda faqatgina, burchak konturdan o'ngga yoki chapga qaraganiga e'tibor bergan holda, $\alpha = 0$ yoki $\alpha = 2\pi$ ni tenglikka qo'yish yetarli.

Foydalanilgan adabiyotlar:

1. Гахов. Ф.Д. Краевые задачи / Ф.Д.Гахов – Москва: Наука, 1977. – 640 с.
2. Лаврентьев М.А., Шабат Б.В. Методы теории функций комплексного переменного / М.А. Лаврентьев, Б.В. Шабат – Москва: Наука, 1973. – 749 с.
3. Панасюк В.В. Распределение напряжений около трещин в пластинах и оболочках / В.В. Панасюк, М.П. Саврук, А.Н. Дацышин. – Киев: Наука. Думка, 1976. – 358 с
4. В.Е. Петрова. Применение интегралов типа Коши при решении краевых задач. / В.Е. Петрова – Воронеж: Наука, 2010. – 44 с