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YOSH TADQIQOTCHI JURNALI

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UCH O'LCHAMLI O'ZGARUVCHAN KOEFFITSIYENTLI INTEGRO-DIFFERENSIAL ISSIQLIK TARQALISH TENGLAMASI UCHUN QO'YILGAN TESKARI MASALANI YECHISH

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ANNOTATSIYA: Ushbu ishda uch o'lchamli o'zgaruvchan koeffitsiyentli integro-differensial issiqlik tarqalish tenglamasi uchun Koshi masala qaraldi. Yangi o'zgaruvchi kiritib, ba'zi almashtirishlardan so'ng o'zgarmas koeffitsiyentli integro-differensial issiqlik tarqalish tenglamasi olindi. O'zgarmas koeffitsiyentli integro-differensial issiqlik tarqalish tenglamasi uchun Koshi masalasi qo'yildi. Masalaning yechimi mavjudligi va yagonaligi uchun asosiy teorema keltirildi.

ABSTRACT: In this paper, the Cauchy problem for a three-dimensional integro-differential heat release equation with variable coefficients is considered. After the introduction of a new variable and some replacements, an integro-differential equation for heat release with a constant coefficient was obtained. The Cauchy problem for the integro-differential equation of heat release with constant coefficients is posed. The main existence and uniqueness theorem for the solution of the problem is given.

Kalit so'zlar: Integro-differensial tenglamalar, Laplas operatori, Koshi masalasi, Volterra ikkinchi tur integral tenglamasi, Gronuolla — Bellman tengsizligi, ketma-ket yaqinlashish usuli

Keywords: Integro-differential equations, Laplace operator, Cauchy problem, Volterra integral equation of the second kind, Gronwall-Bellman inequality, series approximation method

Kirish. $u(x_1, x_2, x_3, t)$, funksiyani $(x) \in \mathbb{R}_T^3$, sohada quyidagi tenglamalardan aniqlash masalasini qaraymiz:

$$u_t - a(t)\Delta u = \int_0^t k(x_1, x_2, \tau) u(x_1, x_2, x_3 t - \tau) d\tau, \quad (x, t) \in \mathbb{R}_T^3, \quad (1)$$

$$u|_{t=0} = \varphi(x_1, x_2, x_3), \qquad (x) \in \mathbb{R}^3, \quad (2)$$

bu yerda $\Delta := \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}$ Laplas operatori

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$$\mathbb{R}_T^3 = \{x, t)|\ (x) \in \mathbb{R}^3,\ 0 < t < T\},$$

$T > 0$ tayinlangan ixtiyoriy son, $a(t) \in E := \{a(t) \in C^1[0, T], 0 < a_0 \leq a(t) \leq a_1 < \infty\}$. Faraz qilaylik $\varphi(x_1, x_2, x_3)$ va $k(x_1, x_2, t)$ funksiyalar uchun quyidagi shartlar bajarilsin:

$$\varphi(x_1, x_2, x_3) \in H^{l+2}(\mathbb{R}^3), \varphi(x_1, x_2, x_3) \geq \varphi_0 = const > 0,\ k(x_1, x_2, t) \in H^{l,l/2}(\overline{\mathbb{R}}_T),$$

bunda

$$\overline{\mathbb{R}}_T = \{(x, t)|\ x \in \mathbb{R}, \qquad 0 \leq t \leq T\}, \qquad l \in (0,1).$$

Berilgan a va k funksiyalar uchun (1) integro-differensial tenglamadan $u(x_1, x_2, x_3, t)$ funksiyani (2) boshlang'ich shart orqali topish masalasiga Koshi masalasi deyiladi. Bunday masalalar teskari masalalar nazariyasida to'g'ri masala deb ataladi. (1) va (2) Koshi masalasining yechimi Volterra tipidagi integral tenglamaga ekvivalent bo'ladi [1-3]. Buning uchun quyidagi formuladan foydalanamiz.

$$p(x, t) = \int_{\mathbb{R}^3} \varphi\,(\xi_1, \xi_2, \xi_3) G(x_1 - \xi_1, x_2 - \xi_2; x_3 - \xi_3; \theta(t)) d\xi_1 d\xi_2 d\xi_3 +$$

$$+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^3} F\left(\xi_1, \xi_2, \xi_3, \theta^{-1}(\tau)\right) \cdot$$

$$G(x_1 - \xi_1, x_2 - \xi_2; x_3 - \xi_3; \theta(t) - \tau) d\xi_1 d\xi_2 d\xi_3. \qquad (3)$$

(3) formula quyidagi o'zgaruvchan koeffitsiyentli issiqlik tarqalish tenglamasi uchun Koshi masalasini yechimini ifodalaydi:

$$p_t - a(t)\Delta p = F(x_1, x_2, x_3, t),\ \ (x) \in \mathbb{R}^3, t > 0,$$

$$p(x_1, x_2, x_3, ,0) = \varphi(x_1, x_2, x_3,),\ \ (x) \in \mathbb{R}^3.$$

(3) da $\theta(t) = \int_0^t a\,(\tau) d\tau$ va $\theta^{-1}(t)$ funksiya $\theta(t)$ funksiyaning teskari funksiyasidir.

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$$G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)=\frac{1}{4\pi(\theta(t)-\tau)}e^{-\frac{|x-\xi_1|^2+|x_2-\xi_2|^2+|x_3-\xi_3|^2}{4(\theta(t)-\tau)}},$$

$\frac{\partial}{\partial t}-a(t)\triangle$ o‘zgaruvchan koeffitsiyentli differensial operatorining fundamental yechim. (3) formuladan foydalanib (1)-(2) masala yechimini quyidagicha ikkinchi tur Volterra integral tenglamasi ko‘rinishida yozamiz:

$$u(x_1,x_2,x_3,t)=\int\limits_{\mathbb{R}^3}\varphi(\xi_1,\xi_2,\xi_3)G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)d\xi_1d\xi_2d\xi_3+$$

$$+\int\limits_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int\limits_{\mathbb{R}^3}\int\limits_0^{\theta^{-1}(\tau)}k(\xi_1,\xi_2,\alpha)u(\xi_1,\xi_2,\xi_3,\theta^{-1}(\tau)-\alpha)\times$$

$$\times G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)d\alpha d\xi_1d\xi_2d\xi_3. \qquad (4)$$

Quyidagi lemma o‘rinli:

Teorema 1. Faraz qilaylik, $\varphi(x_1,x_2,x_3)\in H^{l+2}(\mathbb{R}^3)$, $k(x_1,x_2,t)\in H^{l,l/2}(\bar{\mathbb{R}}_T)$, va $a(t)\in E$ bo‘lsin. U holda (4) integral tenglamaning $H^{l+2,(l+2)/2}(\bar{\mathbb{R}}_T^3)$ sinfga qarashli yagona $u=u(,t)$ yechimi mavjud.

Isbot. Teoremani isbotlash uchun ketma-ket yaqinlashish usulidan foydalanamiz [4-6]. Buning uchun dastlabki qadam sifatida, ushbu

$$u_0(x_1,x_2,x_3,t)=\int\limits_{\mathbb{R}^3}\varphi(\xi_1,\xi_2,\xi_3)G(x_1,-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t))d\xi_1d\xi_2d\xi_3,$$

$$u_1(x_1,x_2,x_3,t)=\int\limits_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int\limits_{\mathbb{R}^3}\int\limits_0^{\theta^{-1}(\tau)}k(\xi_1,\xi_2\alpha)u_0(\xi_1,\xi_2,\xi_3,\theta^{-1}(\tau)-\alpha)\times$$

$$\times G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)d\alpha d\xi_1d\xi_2d\xi_3,$$

$$u_2(x_1,x_2,x_3,t)=\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^3}\int_0^{\theta^{-1}(\tau)} k\,(\xi_1,\xi_2,\alpha)u_1(\xi_1,\xi_2,\xi_3,\theta^{-1}(\tau)-\alpha)\times$$

$$\times\, G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)d\alpha d\xi_1 d\xi_2 d\xi_3,$$

$$\dots\quad \dots\quad \dots\quad \dots\quad \dots\quad \dots\quad \dots$$

umumiy hadi sifatida esa

$$u_j(x_1,x_2,x_3,t)=\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^3}\int_0^{\theta^{-1}(\tau)} k\,(\xi_1,\xi_2,\alpha)u_{j-1}(\xi_1,\xi_2,\xi_3,(\tau)-\alpha)\times$$

$$\times\, G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)d\alpha d\xi_1 d\xi_2 d\xi_3,\quad (x_1,x_2,x_3,t)\in\mathbb{R}_T^3,$$

$$k=1{,}2\,{,}3 \qquad (5)$$

integralni qaraymiz.

Parabolik tipga tegishli tenglamalar uchun qo‘yilgan Koshi masalasinig umumiy nazariyasiga ko‘ra, (5) tenglik bilan aniqlangan barcha $u_j(x_1,x_2,x_3,t)$ funksiyalar $\mathbb{R}_T^3$ da korrekt hisoblanadi. $\varphi_0=|\varphi(x_1,x_2,x_3)|^l$ belgilashdan foydalanib $\mathbb{R}_T^3$ da (1.1.5) integral tenglama yordamida aniqlangan $u_j(x_1,x_2,x_3,t)$ funksiyalarni modul jihatdan baholaymiz:

$$|u_0(x_1,x_2,x_3,t)|_T^{l+2,(l+2)/2}\le\varphi_0,$$

Xuddi shunday

$$|u_1(x_1,x_2,x_3,t)|_T^{l+2,(l+2)/2}\le\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^3}\int_0^{\theta^{-1}(\tau)} |\,k(\xi_1,\xi_2,\alpha)|_T^{l,l/2}\times$$

$$\times\,|u_0(\xi_1,\xi_2,\xi_3,\theta^{-1}(\tau)-\alpha)|_T^{l+2,\frac{l+2}{2}}G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)dad\xi_1 d\xi_2 d\xi_3\le$$

$$\leq \varphi_0 \frac{a_1 k_0 T}{a_0} \frac{t}{1!},$$

bu yerda $k_0 := |k(\xi_1, \xi_2, \alpha)|_T^{l,l/2}$. Baholashni $u_2(x_1, x_2, x_3, t)$ uchun ham amalga oshiramiz:

$$|u_2(x_1, x_2, x_3, t)|_T^{l+2,(l+2)/2} \leq \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^3} \int_0^{\theta^{-1}(\tau)} |\, k(\xi_1, \xi_2, \alpha)|_T^{l,l/2} \times$$

$$\times\, |u_1(\xi_1, \xi_2, \xi_3, \theta^{-1}(\tau) - \alpha)|_T^{l+2,\frac{l+2}{2}} G(x_1 - \xi_1, x_2 - \xi_2; x_3 - \xi_3;\;\; \theta(t) - \tau) d\alpha d\xi_1 d\xi_2 d\xi_3 \leq$$

$$\leq \varphi_0 (\frac{a_1 k_0 T}{a_0})^2 \frac{t^2}{2!},$$

… … … … … … …

$$|u_j(x_1, x_2, x_3, t)|_T^{l+2,(l+2)/2} \leq \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^3} \int_0^{\theta^{-1}(\tau)} |\, k(\xi_1, \xi_2, \alpha)|_T^{l,l/2} \times$$

$$\times\, |u_{j-1}(\xi_1, \xi_2, \xi_3, \theta^{-1}(\tau) - \alpha)|_T^{l+2,\frac{l+2}{2}} G(x_1 - \xi_1, x_2 - \xi_2; x_3 - \xi_3; \theta(t) - \tau) d\alpha d\xi_1 d\xi_2 d\xi_3 \leq$$

$$\leq \varphi_0 (\frac{a_1 k_0 T}{a_0})^j \frac{t^j}{j!},$$

… … … … … … …

yuqoridagi bahoda biz quyidagi tenglikdan foydalandik:

$$\int_{\mathbb{R}^3} G\,(x_1 - \xi_1, x_2 - \xi_2; x_3 - \xi_3; \theta(t) - \tau) d\xi_1 d\xi_2 d\xi_3 = 1 \qquad (6)$$

yuqoridagi olingan baholardan foydalanib, quyidagi funksional qatorni tuzib olamiz:

$$\sum_{j=0}^{\infty} u_j\,(x_1, x_2, x_3, t)$$

hosil bo‘lgan funksional qatorni $(x,t) \in \mathbb{R}_T^3$ sohada sonli qator bilan mojarantlaymiz.

$$\sum_{j=0}^{\infty} |\, u_j(x_1,x_2,x_3,t)| \leq \sum_{j=0}^{\infty} \varphi_0\, (\frac{a_1k_0T}{a_0})^j \frac{t^j}{j!} \leq \varphi_0 \sum_{j=0}^{\infty} (\frac{a_1k_0T^2}{a_0})^j \frac{1}{j!}$$

sonli qatorni ma’lum alomatlaridan foydalanib yaqinlashuvchi ekanligini ko‘rsatamiz. Buning uchun sonli qatorning yaqinlashishning Dalamber alomatidan foydalanib

$$a_j = (\frac{a_1k_0T^2}{a_0})^j \frac{1}{j!}, a_{j+1} = (\frac{a_1k_0T^2}{a_0})^{j+1} \frac{1}{j+1!}$$

ekanligidan

$$\frac{a_{j+1}}{a_j} = (\frac{a_1k_0T^2}{a_0})^{j+1} \frac{1}{j+1!} \cdot (\frac{a_0}{a_1k_0T^2})^j j! = \frac{a_1}{a_0} k_0T^2 \frac{1}{(j+1)}$$

bo‘ladi. Ravshanki

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = 0.$$

Demak, hosil qilingan sonli qator yaqinlashuvchi. Funksional qatorning tekis yaqinlashish haqidagi Veyershtrass teoremasiga ko‘ra olingan funksional qator tekis yaqinlashuvchiligi kelib chiqadi [7,8]. Bu natijadan (4) integral tenglama yordamida aniqlangan $u_j(x_1,x_2,x_3,t)$ funksiyalar ketma-ketligi $H^{l+2,(l+2)/2}(\mathbb{R}_T^3)$ funksiyalar fazosida aniqlangan biror $u(x_1,x_2,x_3,t)$ funksiyaga tekis yaqinlashadi. Shunday qilib (1)-(2) Koshi masalasinig $H^{l+2,(l+2)/2}(\mathbb{R}_T^3)$ sinfga tegishli yechimi mavjudligini ko‘rsatdik.

Biz hozircha (4) integral tenglama yechimga ega ekanligini ko‘rdik. Endi (4) tenglama yagona yechimga ega ekanligini ko‘rsatamiz. Buning uchun teskarisiga faraz qilaylik, ya’ni (4) integral tenglama ikkita aynan teng bo‘lmagan $u^1(x_1,x_2,x_3,t)$ va $u^2(x_1,x_2,x_3,t)$ yechimlarga ega bo‘lsin [9]:

$$u_0(x_1,x_2,x_3,t)=\int\limits_{\mathbb{R}^3}\varphi\,(\xi_1,\xi_2,\xi_3)G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)d\xi_1d\xi_2d\xi_3,$$

$$u^1(x_1,x_2,x_3,t)=\int\limits_{\mathbb{R}^3}\varphi\,(\xi_1,\xi_2,\xi_3)G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t))d\xi_1d\xi_2d\xi_3+$$

$$+\int\limits_0^{\theta(t)}\frac{d\tau}{a\big(\theta^{-1}(\tau)\big)}\int\limits_{\mathbb{R}^3}\int\limits_0^{\theta^{-1}(\tau)}k\,(\xi_1,\xi_2,\alpha)u_1(\xi_1,\xi_2,\xi_3,\theta^{-1}(\tau)-\alpha)\times$$

$$\times\,G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3\theta(t)-\tau)d\alpha d\xi_1d\xi_2d\xi_3$$

va

$$u^2(x_1,x_2,x_3,t)=\int\limits_{\mathbb{R}^3}\varphi\,(\xi_1,\xi_2,\xi_3)G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t))d\xi_1d\xi_2d\xi_3+$$

$$+\int\limits_0^{\theta(t)}\frac{d\tau}{a\big(\theta^{-1}(\tau)\big)}\int\limits_{\mathbb{R}^3}\int\limits_0^{\theta^{-1}(\tau)}k\,(\xi_1,\xi_2,\alpha)u_2(\xi_1,\xi_2,\xi_3,\theta^{-1}(\tau)-\alpha)\times$$

$$\times\,G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)dad\xi_1d\xi_2d\xi_3$$

$u^1(x_1,x_2,x_3,t)$ va $u^2(x_1,x_2,x_3,t)$ yechimlar ayirmasini qarab, ularning farqini $Z(x_1,x_2,x_3t)=u^1(x_1,x_2,x_3,t)-u^2(x_1,x_2,x_3,t)$ bilan belgilaymiz. Natijada

$$Z(x_1,x_2,x_3,t)=\int\limits_0^{\theta(t)}\frac{d\tau}{a\big(\theta^{-1}(\tau)\big)}\int\limits_{\mathbb{R}^3}\int\limits_0^{\theta^{-1}(\tau)}k\,(\xi_1,\xi_2,\alpha)Z(\xi_1,\xi_2,\xi_3,\theta^{-1}(\tau)-\alpha)\times$$

$$\times\,G(x_1-\xi_1,x_2-\xi_2;x_3-\xi_3;\theta(t)-\tau)dad\xi_1d\xi_2d\xi_3, \qquad (7)$$

bir jinsli ikkinchi tur Volterra integral tenglamasini olamiz. Har bir tayinlangan $t\in[0,T]$ da $(x)\in\mathbb{R}^3$ bo‘yicha $Z(x_1,x_2,x_3,t)$ funksiyaning modul jihatdan supremumini $\tilde{Z}(t)$ orqali belgilaymiz, ya’ni:

$$\tilde{Z}(t) = \sup_{x\in\mathbb{R}^3} |Z(x_1, x_2, x_3, t)|, \qquad t \in [0, T].$$

U holda (7) integral tenglamadan

$$\tilde{Z}(t) \le \frac{a_1 k_0 T}{a_0} \int_0^{a_1 t} \tilde{Z}(\tau) d\tau, \qquad t \in [0, T].$$

integral tengsizlik olinadi. Gronuolla — Bellman tengsizligiga koʻra, oxirgi integral tengsizlik faqat $\tilde{Z}(t) = 0$ yechimga ega. Bundan esa $\bar{\mathbb{R}}_T^3$ sohada $Z(x_1, x_2, x_3, t) = 0$ yoki $u^1(x_1, x_2, x_3, t) = u^2(x_1, x_2, x_3, t)$ ekanligi kelib chiqadi. Shunday qilib (4) integral tenglama yagona yechimga ega ekan. **Teorema isbotlandi.**

MUHOKAMA

Uch oʻlchamli oʻzgaruvchan koeffitsiyentli issiqlik tarqalish integro–differensial tenglamasi uchun qo'yilgan Koshi masalasining bir qiymatli yechiluvchanligini koʻrsatish metodlardan nolokal shartlar yordamida qo'yilgan Koshi masalasini yechishda foydalanilsa bo'ladi. **Bundan tashqari,** issiqlik tarqalish integro–differensial tenglamalari uchun **qo'yilgan tog'ri masalalarni yechishda yuqoridagi natijalardan foydalanilsa maqsadga muvoffiq bo'ladi.**

XULOSA

Uch oʻlchamli oʻzgaruvchan koeffitsiyentli integro-differsial issiqlik tarqalish tenglamasi uchun Koshi masalasi qo'yildi, m**asalaning yechimi mavjudligi va yagonaligi uchun asosiy teorema isbotlandi. Teoremani isbotlashda ketma-ket yaqinlashish (Pikar) usulidan foydalanildi,** Gronuolla — Bellman tengsizligiga asosan **yagonaligi isbotlandi.**

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